Int. J. Heat Mass Transfer. Vol. 27, No. 11, pp. 2111–2117, 1984
Printed in Great Britain

0017-9310/84 $3.00 + 0.00
Pergamon Press Ltd.

Heat transfer and diffusion from the sources with random components

A. Z. DOLGINOV and N. A. SILANT'EV
A. F. Ioffe Physicotechnical Institute, Leningrad, U.S.S.R.

(*Received* 27 *April* 1983)

Abstract—Heat transfer in a medium with a randomly heated surface is considered. Explicit expressions for temperature correlations at different points of the medium are derived. Analogous results are obtained for the problem of diffusion in the case of sources with random components. In many important cases the correlations decrease with increasing distance from the surface as the fourth power of the depth.

1. INTRODUCTION

IN SPITE of the fact that the problems of heat conduction and diffusion theories have already become the subjects of mathematical rather than theoretical physics, there are a number of important physical problems which have been left nearly untouched by investigators. These include the problems of heat transfer in the case of the sources having components which randomly fluctuate in space and time. The physical importance of the problems is obvious, e.g. in situations when a rather uniform heating is required or estimations of the size and lifetime of the regions with temperatures differing from the average one are needed.

There are similar diffusion problems in which the sources have random components.

In contrast to most publications on the heat transfer theory where only the average values of temperature are considered, this paper will present the temperature correlations at different space–time points of the medium. The problems considered can be solved with the help of Green's functions. Instead, in this work the equations have been solved directly, since this method is much simpler and the results obtained are more general.

2. HEAT TRANSFER IN A MEDIUM WITH A STOCHASTICALLY HEATED SURFACE

Consider a homogeneous flat layer of the medium which is stationarily heated from one side and which releases heat from the other. Let the stationary heating conditions be characterized by (a) the time-averaged temperature of the surface and by (b) the correlation function of temperature fluctuations at different points on the surface.

Represent the temperature as a sum of the averaged and fluctuating components $T = T_0 + T_1$, where $T_0 = \langle T \rangle$, $\langle T_1 \rangle = 0$. The angular brackets, $\langle\ \rangle$, denote the averages over an ensemble of the stochastic parameters of the problem.

The temperature correlation function at points r_1 and r_2 at times t_1 and t_2 will depend on the coordinates z_1 and z_2 measured along the layer depth, on the distance $r = |x_1 - x_2|$ between the points in the direction normal to the height, and on the time interval $\tau = t_1 - t_2$. Denote the temperatures at the corresponding points by $T(1)$ and $T(2)$, respectively, and the correlation function by

$$\langle T_1(1)T_1(2)\rangle \equiv B(12) = B(z_1; z_2; r; \tau). \quad (1)$$

The heat conduction equation is [1]

$$\frac{\partial T(\mathbf{r}; t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r}; t). \quad (2)$$

The function $T_0(\mathbf{r})$ ($\langle T_0^2 \rangle = T_0^2$, $\langle T_0 T_1 \rangle = 0$) is regarded to be known from equation (2). Now, take the sum and the difference of two equations (2) for points 1 and 2, multiplied by $T_1(2)$ and $T_1(1)$, respectively, and perform the averaging. This will yield two equations for the correlation functions

$$\frac{\partial}{\partial \tau} B(12) = \frac{\kappa}{2}[\nabla^2(1) - \nabla^2(2)]B(12)$$
$$[\nabla^2(1) + \nabla^2(2)]B(12) = 0. \quad (3)$$

It is convenient to express the solution for $B(12)$ in the form

$$B(z_1; z_2; r; \tau) = \int_0^\infty B(z_1; z_2; \lambda; \tau) J_0(\lambda r)\lambda \, d\lambda \quad (4)$$

where $J_0(\lambda r)$ is the Bessel function. This will give

$$\frac{\partial}{\partial \tau} B(z_1; z_2; \lambda; \tau) = \frac{\kappa}{2}(\partial^2/\partial z_1^2 - \partial^2/\partial z_2^2) B(z_1; z_2; \lambda; \tau) \quad (5)$$
$$(\partial^2/\partial z_1^2 + \partial^2/\partial z_2^2 - 2\lambda^2) B(z_1; z_2; \lambda; \tau) = 0.$$

Using the Fourier expansion

$$B(z_1; z_2; \lambda; \tau) = \int_0^\infty [\cos \omega\tau B_c(z_1; z_2; \lambda; \omega) + \sin \omega\tau B_s(z_1; z_2; \lambda; \omega)] \, d\omega \quad (6)$$

NOMENCLATURE

A_1, A_2, A_3, A_4	coefficients in equation (8)
$B(12)$	temperature correlator at points 1 and 2, $B(z_1; z_2; r; \tau)$
$b_c(\lambda; \omega)$	Bessel and Fourier transformations of the temperature correlator $B(0; 0; r; \tau)$ on the lower surface ($z_1 = z_2 = 0$)
c	specific heat
$c(r; t)$	concentration of substance
$C(x)$	Fresnel's integral
D	diffusion coefficient
$E(k)$	elliptic integral of the second kind
F	heat flux
$f_c(\lambda; \omega)$	Bessel and Fourier transformations of heat flux correlator on upper surface ($z_1 = z_2 = L$)
$g_c(\lambda; \omega)$	Bessel and Fourier transformations of temperature correlator on upper surface
G	Green's functions
h	constant in equation (10)
k_i	dimensionless roots of characteristic equation [see equation (9)], $r_0 s_i$
L	thickness of plane-parallel layer
p	dimensionless variable of Bessel transformation, $r_0\lambda$
Q_n	integrals in equation (21)
r_i	radius vector of ith point
r	horizontal distance between points 1 and 2, $\lvert x_1 - x_2 \rvert$
r_0	characteristic length of surface heating
R	dimensionless horizontal distance, r/r_0
$S(x)$	Fresnel's integral
$s_{1,2}$	roots of characteristic equation
T	temperature
T_0	mean temperature
T_1	fluctuating temperature
t_1, t_2	time points
t	dimensionless time difference, $\tau/\tau_0 = (t_1 - t_2)/\tau_0$
x	horizontal coordinate of radius vector $\mathbf{r}$
z	vertical coordinate of radius vector $\mathbf{r}$
Z	dimensionless vertical coordinate, z/r_0.

Greek symbols

α	dimensionless parameter, $r_0^2/\kappa\tau_0$
$\delta(x)$	Dirac's δ-function
$\Delta, \Delta_1, \Delta_2$	coefficients in equations (11), (13), and (14)
κ	thermal diffusivity
λ	variable in Bessel transformation
$\mu(x)$	Green's function [equation (29)]
ν	dimensionless frequency, $\omega\tau_0$
ρ	density
σ	Stephan–Boltzmann constant
τ	time difference, $t_1 - t_2$
ω	frequency in Fourier transformation.

one obtains

$$\left(\frac{\partial^2}{\partial z_i^2}\right) B_c(z_1; z_2; \lambda; \omega) - \lambda^2 B_c(z_1; z_2; \lambda; \omega) = \pm \frac{\omega}{\kappa} B_s(z_1; z_2; \lambda; \omega)$$

$$\left(\frac{\partial^2}{\partial z_i^2} - \lambda^2\right) B_s(z_1; z_2; \lambda; \omega) = \mp \frac{\omega}{\kappa} B_c(z_1; z_2; \lambda; \omega), \quad i = 1, 2. \tag{7}$$

The upper signs on the RHS of equation (7) relates to $i = 1$ and the lower, to $i = 2$. The elimination of B_c or B_s yields

$$\begin{aligned} B_s(z_1; z_2; \lambda; \omega) = {} & (A_1 \exp[s_1(z_1+z_2)] \\ & + A_2 \exp[-s_1(z_1+z_2)]) \sin s_2(z_1-z_2) \\ & + (A_3 \cos s_2(z_1+z_2) + A_4 \sin s_2(z_1+z_2)) \\ & \times \sinh s_1(z_1-z_2). \end{aligned} \tag{8}$$

The expression for B_c can be derived from that for B_s by replacing sin ⇄ cos and sinh ⇄ cosh and by changing the signs in front of A_1 and A_4. This provides the obvious relation $B(z_1; z_2; r; \tau) = B(z_2; z_1; r; -\tau)$.

The roots of the characteristic equation are defined as

$$s_{1,2} = \sqrt{\left(\frac{\sqrt{(\lambda^4 + \omega^2/\kappa^2)} \pm \lambda^2}{2}\right)} \tag{9}$$

where the upper sign under the radical corresponds to s_1 and the lower, to s_2. The coefficients A_i are to be determined from the boundary conditions.

The boundary values of temperature and its correlators are considered to be known from the heating conditions on the surface. The average heat flux propagating through the slab is also regarded as known.

Depending on the physical conditions, the following variables are to be fixed on the upper boundary: (a) the temperature and its correlators, (b) the heat flux and its correlators, or (c) the temperature dependence of the heat flux (radiation condition) together with a corresponding relation between the flux and temperature correlators.

Condition (c) leads to the well-known difficulties connected with the nonlinear dependence ($F = \sigma T^4$) of the flux F on temperature. However, in the majority of cases the fluctuations of the emerging flux are much below its average value F_0. This allows one to set $F = F_0+F_1$, where $F_1 \ll F_0$, to define the effective temperature T_e on the upper layer from the condition $F_0 = \sigma T_e^4$ and to use the approximation $F_1 \approx F_0(4T/T_e-4)$. Then the boundary conditions for case (c) can be expressed as

$$\frac{\partial}{\partial z_1} B_{c,s}|_{z_1=L} = -hB_{c,s}|_{z_1=L}, \qquad B_c|_{z_1=z_2=0} = b_c \tag{10}$$

where $h = 4\sigma T_e^3 \kappa \rho c$, κ is the thermal diffusivity, ρ the density of the medium, c the specific heat, and σ the Stephan–Boltzmann constant. In this case one obtains

$$\begin{aligned} A_1 &= -b_c\Delta^{-1}[(s_1-h)^2+s_2^2] \exp(-2s_1L) \\ A_2 &= b_c\Delta^{-1}[(s_1+h)^2+s_2^2] \exp(2s_1L) \\ A_3 &= 2b_c\Delta^{-1}[(s_1^2+s_2^2-h^2) \sin 2s_2L-2hs_2 \cos 2s_2L] \\ A_4 &= -2b_c\Delta^{-1}[(s_1^2+s_2^2-h^2) \cos 2s_2L + 2hs_2 \sin 2s_2L] \end{aligned} \tag{11}$$

$$\Delta = 2(s_1^2+s_2^2-h^2) \cos 2s_2L+4hs_2 \sin 2s_2L + 4hs_1 \sinh 2s_1L+2(s_1^2+s_2^2+h^2) \cosh 2s_1L.$$

For a very thick layer ($L\to\infty$), A_1, A_3, $A_4 \to 0$, $A_2 \to b_c(\lambda;\omega)$.

If the temperature correlators on both slab surfaces are known (similarly to the first-kind boundary conditions) and the statistical independence of these correlators on different surfaces is assumed, i.e. $B(0;L;\lambda;\omega) = 0$, then

$$B_c(z_1;z_2;\lambda;\omega)|_{z_1=z_2=0} = b_c(\lambda;\omega), \qquad B_c(z_1;z_2;\lambda;\omega)|_{z_1=z_2=L} = g_c(\lambda;\omega) \tag{12}$$

$$\begin{aligned} A_1 &= \Delta_1^{-1}(-g_c+b_c \exp(-2s_1L)) \\ A_2 &= \Delta_1^{-1}(g_c+b_c \exp(2s_1L)) \\ A_3 &= -2b_c\Delta_1^{-1} \sin 2s_2L \\ A_4 &= 2\Delta_1^{-1}(g_c+b_c \cos 2s_2L) \\ \Delta_1 &= 2(\cosh 2s_1L-\cos 2s_2L). \end{aligned} \tag{13}$$

When the correlations of temperature are given on the lower surface and the correlations of fluxes are given on the upper surface and these are statistically independent (i.e. $\partial B/\partial z_1|_{z_1=0,z_2=L} = 0$), then

$$B_c(z_1;z_2;\lambda;\omega)|_{z_1=z_2=0} = b_c(\lambda;\omega), \qquad \frac{\partial^2}{\partial z_1\,\partial z_2}B(z_1;z_2;\lambda;\omega)|_{z_1=z_2=L} = f_c(\lambda;\omega) \tag{14}$$

$$\begin{aligned} A_1 &= -\Delta_2^{-1}(b_c \exp(-2s_1L)+f_c(s_1^2+s_2^2)^{-1}) \\ A_2 &= \Delta_2^{-1}(b_c \exp(2s_1L)+f_c(s_1^2+s_2^2)^{-1}) \\ A_3 &= 2b_c\Delta_2^{-1} \sin 2s_2L \\ A_4 &= 2\Delta_2^{-1}(-b_c \cos 2s_2L+f_c(s_1^2+s_2^2)^{-1}) \\ \Delta_2 &= 2(\cosh 2s_1L+\cos 2s_2L). \end{aligned} \tag{15}$$

It can be easily seen that for $h\to\infty$ (which is equivalent to $B_c = 0$ on the upper surface), conditions (10) convert into conditions (12) with $g_c = 0$. In the reverse case of $h\to 0$, conditions (10) go over into conditions (14) with $f_c = 0$.

The considered basic boundary conditions for correlators $B(z_1;z_2;r;\tau)$ can be extended to other cases. In particular, one could easily consider the case of dependent conditions on the surfaces.

In order to find the correlator $B(z_1;z_2;r;\tau)$, it is necessary that the values of $B_c(z_1;z_2;\lambda;\omega)$ and $B_s(z_1;z_2;\lambda;\omega)$, found from the boundary conditions, be inserted into equation (6) and that then integral (4) be evaluated.

3. A SEMI-INFINITE MEDIUM

Consider the distribution of temperature fluctuations within a semi-infinite medium heated at its boundary. The inhomogeneity and nonuniformity of heating are specified by the correlator on the surface $B_c(0;0;r;\tau)$

$$B_c(0;0;r;\tau) = \int_0^\infty d\lambda\, \lambda \int_0^\infty d\omega \cos \omega\tau J_0(\lambda r) b_c(\lambda;\omega). \tag{16}$$

Then, according to the above results, $A_1 = A_3 = A_4 = 0$ and $A_2 = b_c(\lambda;\omega)$. It is convenient to introduce the dimensionless quantities

$$R = r/r_0, \quad t = \tau/\tau_0, \quad k_i = r_0 s_i, \quad p = r_0\lambda, \quad \nu = \omega\tau_0, \quad Z = z/r_0, \quad \alpha = r_0^2/\kappa\tau_0 \tag{17}$$

where r_0 and τ_0 are the characteristic coordinate and time scales of temperature correlations at the medium boundary. In these dimensionless variables, one has

$$B(Z_1;Z_2;R;t) = \int_0^\infty d\nu \int_0^\infty dp\, pJ_0(pR) \times \cos[\nu t-k_2(Z_1-Z_2)] \cdot b_c(p;\nu) \times \exp[-k_1(Z_1+Z_2)] \tag{18}$$

$$b_c(p;\nu) = \frac{2}{\pi}\int_0^\infty dt \int_0^\infty dR\, RJ_0(pR) \cos \nu t B_c(0;0;R;t).$$

It is also convenient to introduce the Green's function which determines the correlator at any point inside the medium through the correlator on the surface

$$B(Z_1;Z_2;R;t) = \int_0^\infty dt' \int_0^\infty dR'\, R'G \times (Z_1;Z_2;R;t;R';t') B_c(0;0;R';t')$$

$$G(Z_1;Z_2;R;t;R';t') = \frac{2}{\pi}\int_0^\infty d\nu \int_0^\infty dp\, pJ_0(pR) \times J_0(pR') \cos \nu t' \cos[\nu t-k_2(Z_1-Z_2)] \times \exp[-k_1(Z_1+Z_2)]. \tag{19}$$

In the case of large depths, when $Z_1+Z_2 \gg 1$ for sufficiently close points

$$(|Z_1-Z_2| \ll (Z_1+Z_2), \quad R^2+R'^2 \ll (Z_1+Z_2)^2, \quad t+t' \ll \alpha(Z_1+Z_2)^2)$$

the Green's function has the following asymptotic form

$$G(Z_1;Z_2;R;t;R';t') \approx \frac{32}{\pi\alpha(Z_1+Z_2)^4} \times \left\{1-\frac{4(Z_1-Z_2)^2+3R^2+3R'^2}{(Z_1+Z_2)^2} + \frac{96t(Z_1-Z_2)}{\alpha(Z_1+Z_2)^3} + \frac{1}{(Z_1+Z_2)^4} \times \left[6R^4+6R'^4+24R^2R'^2-672\frac{t^2+t'^2}{\alpha^2} + 9(Z_1-Z_2)^4+15(Z_1-Z_2)^2(R^2+R'^2)\right]\right\}. \quad (20)$$

This expression allows one to find $B(Z_1;Z_2;R;t)$ at $Z_1+Z_2 \gg 1$ only for such correlators $B_c(0;0;R;t)$ for which the following integrals ($n=0,2; m=0,2,4$) have the finite values

$$Q = \int_0^\infty \mathrm{d}R\, R \int_0^\infty \mathrm{d}t\, B_c(0;0;R;t),$$
$$Q_{m,n} = \int_0^\infty \mathrm{d}R\, R \int_0^\infty \mathrm{d}t\, t^n R^m B_c(0;0;R;t). \quad (21)$$

Thus, for the Gaussian correlator

$$B_c(0;0;R;t) = B_0 \exp(-R^2)\exp(-t^2) \quad (22)$$

one obtains $Q=\sqrt{\pi}B_0/4$, $Q_{0,2}=\sqrt{\pi}B_0/8$, $Q_{2,0}=\sqrt{\pi}B_0/4$, $Q_{4,0}=\sqrt{\pi}B_0/2$ and equation (19) acquires the form

$$B(Z_1;Z_2;R;t) \approx \frac{8B_0}{\sqrt{\pi}\alpha(Z_1+Z_2)^4} \times \left\{1-\frac{4(Z_1-Z_2)^2+3R^2+3}{(Z_1+Z_2)^2}+\frac{96t(Z_1-Z_2)}{\alpha(Z_1+Z_2)^3} + \frac{1}{(Z_1+Z_2)^4}\left[6R^4+24R^2+12-672\frac{t^2+0.5}{\alpha^2} + 9(Z_1-Z_2)^4+15(Z_1-Z_2)^2(1+R^2)\right]\right\}. \quad (23)$$

The power-law dependence for the damping of correlation far from the surface is valid only for the sign-positive correlators $B_c(0;0;T;t)$. The correlator $\langle T_1(t)T_1(t+\tau)\rangle$ is positive if the correlation time τ_c is much smaller than the characteristic time-scale τ_1 of the heating regime. This is also true for space correlation. But if $\tau_c \gtrsim \tau_1$, the sign of the correlator is variable and the thermal waves of subsequent heating and cooling compensate each other very effectively thus leading to an exponential damping. For example, for the following correlator on the surface

$$B_c(0;0;R;t) = B_0 J_0(p_0 R)\cos \nu_0 t \quad (24)$$

the correlation inside the medium has the form

$$B(Z_1;Z_2;R;t) = B_0 J_0(p_0R)\cos[\nu_0 t - k_2(\nu_0;p_0)(Z_1-Z_2)] \times \exp[-(Z_1+Z_2)k_1(\nu_0;p_0)]. \quad (25)$$

In this case the initial inhomogeneities of heating decrease exponentially within the slab and undergo oscillations in space and time.

The Green's function $G(Z_1;Z_2;R;t;R';t')$ for $R'=0$, $t'=0$ describes the propagation of perturbations called 'white noise'. This case corresponds to the formulae

$$B_c(0;0;R;t) = \frac{B_0}{R}\delta(t)\delta(R),$$
$$B(Z_1;Z_2;R;t) = \frac{B_0}{2}G(Z_1;Z_2;R;t;0;0). \quad (26)$$

We see that at some distance from the surface the correlation reaches a maximum because the thermal inhomogeneities tend to be smoothed, with the smoothing time being smaller than the time for correlation damping.

Note that any distribution $B_c(0;0;R;t)$ with the finite integral Q at a sufficiently large distance of the points considered from the medium boundary leads to the correlator of the form of equations (26) corresponding to the white noise in the case of $B_0=2Q$.

If the correlator $B_c(0;0;R;t)$ is equal to $B_c(0;0;|t|)$, i.e. the temperature change on the surface is purely temporal, then the function $B(Z_1;Z_2;R;t)$ is independent of R. In this case

$$G(Z_1;Z_2;t;t') = \int_0^\infty \mathrm{d}R'\, R' G(Z_1;Z_2;R;t;R';t') = \frac{2}{\pi}\int_0^\infty \mathrm{d}\nu \cos \nu t' \cos\left[\nu t - \sqrt{\left(\frac{\alpha\nu}{2}\right)}(Z_1-Z_2)\right] \times \exp\left[-\sqrt{\left(\frac{\alpha\nu}{2}\right)}(Z_1+Z_2)\right]. \quad (27)$$

In the general case, when $Z_1 \neq Z_2$, this Green's function is expressed through the Fresnel integral in a complex plane. Restricting the discussion to a somewhat simplified case $Z_1 = Z_2 \equiv Z$ results in

$$G(Z;t;t') = \frac{1}{\pi\alpha Z^2}\left[\mu\left(\frac{2|t-t'|}{\alpha Z^2}\right)+\mu\left(\frac{2|t+t'|}{\alpha Z^2}\right)\right] \quad (28)$$

$$\mu(x) = \frac{2\sqrt{(2\pi)}}{x^{3/2}}\left[\cos\left(\frac{1}{x}\right)\left(\frac{1}{2}-C\left(\frac{1}{x}\right)\right) + \sin\left(\frac{1}{x}\right)\left(\frac{1}{2}-S\left(\frac{1}{x}\right)\right)\right] \quad (29)$$

$$C(x) = \int_0^x \frac{\cos t\, \mathrm{d}t}{\sqrt{(2\pi t)}}, \quad S(x) = \int_0^x \frac{\sin t\, \mathrm{d}t}{\sqrt{(2\pi t)}}. \quad (30)$$

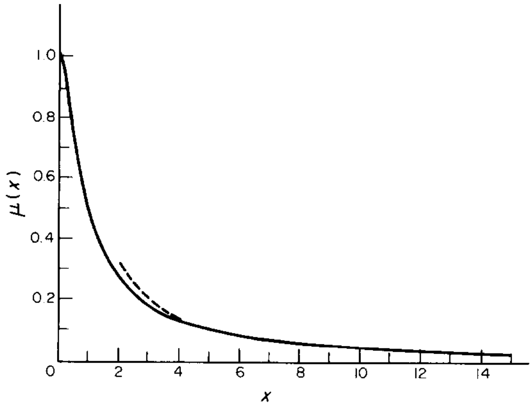

FIG. 1. Values of the Green's function $\mu(x)$. The dashed line denotes the values obtained from asymptotic formula (33).

The correlator $B(Z;t)$ can be expressed as

$$B(Z;t) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}x\ \mu(x)\left[B_c\left(0;0;\left|t+\frac{\alpha Z^2 x}{2}\right|\right) + B_c\left(0;0;\left|t-\frac{\alpha x Z^2}{2}\right|\right)\right]. \quad (31)$$

The function $\mu(x)$ is presented in Fig. 1. At $x \ll 1$, the following asymptotic formula holds

$$\mu(x) \approx 1 - 15x^2/4, \quad x \ll 1. \quad (32)$$

At $x \gg 1$

$$\mu(x) \approx \sqrt{(2\pi)}x^{-3/2}\left(1 - 2\sqrt{\left(\frac{2}{\pi x}\right)} + \frac{1}{x}\right), \quad x \gg 1. \quad (33)$$

This formula is valid up to $x = 5$.

In the case of $z^2 \gg \kappa\tau$, the correlator $B(z;\tau)$ acquires the form

$$B(z;\tau) \approx \frac{2\kappa\tau_0}{\pi z^2} Q_0\left(1 - \frac{15\kappa^2\tau^2}{z^4} - \frac{15\kappa^2}{z^4}\cdot\frac{Q_2}{Q_0}\right) \quad (34)$$

$$Q_n = \int_0^\infty \mathrm{d}t\ t^n B_c(0;0;t). \quad (35)$$

It can be seen that the absence of spatial temperature fluctuations on the surface of the medium leads to a much slower decrease of correlation with the distance from the surface than in the general case, equation (23).

If $2\kappa\tau \gg z^2$, but $2\kappa\tau_0 \ll z^2$, then ($z_0^2 = 2\kappa\tau_0$)

$$B(z;\tau) \approx \frac{Q_0\tau_0 z}{\sqrt{(\pi\kappa\tau^3)}} \equiv Q_0\sqrt{\left(\frac{2}{\pi}\right)}\frac{z}{z_0}\left(\frac{\tau_0}{\tau}\right)^{3/2}, \qquad \frac{\tau}{\tau_0} \gg \left(\frac{z}{z_0}\right)^2 \gg 1. \quad (36)$$

This case corresponds to the correlation in the zone of the temperature waves propagating from the source that acts during the time $\sim \tau_0$ which is of the order of the correlation time of the surface fluctuation. Inside this zone there is a region of temperature increase. Of course, far from the front of the temperature wave ($z^2 \gg \kappa\tau$), there is always the region of decreasing temperature (see Fig. 2).

In the region with $z \ll r_0$ and $R \ll r_0$, where r_0 is the characteristic correlation scale on the surface, it is possible to assume that $B(z;R;t)$ is independent of R.

Consider now the case when the correlator $B_c(0;0;R;t)$ is independent of t, i.e. the temperature correlation is purely spatial. This is approximately true if $\tau \ll \tau_0$. In this case $B_c(0;0;R;t) = B_c(0;0;R)$ and $B(Z_1;Z_2;R;t) = B(Z_1+Z_2;R)$

$$B(Z_1+Z_2;R) = \int_0^\infty \mathrm{d}R'\ R'G \times (Z_1+Z_2;R;R')B_c(0;0;R') \quad (37)$$

$$G(Z_1+Z_2;R;R') = \int_0^\infty \mathrm{d}t'\ G(Z_1;Z_2;R;t;R';t') = \int_0^\infty \mathrm{d}p\ pJ_0(pR)J_0(pR')\,\mathrm{e}^{-p(Z_1+Z_2)} = \frac{2}{\pi}\cdot\frac{Z_1+Z_2}{\sqrt{((Z_1+Z_2)^2+(R+R')^2)}} \times \frac{E(k)}{(Z_1+Z_2)^2+(R-R')^2} \quad (38)$$

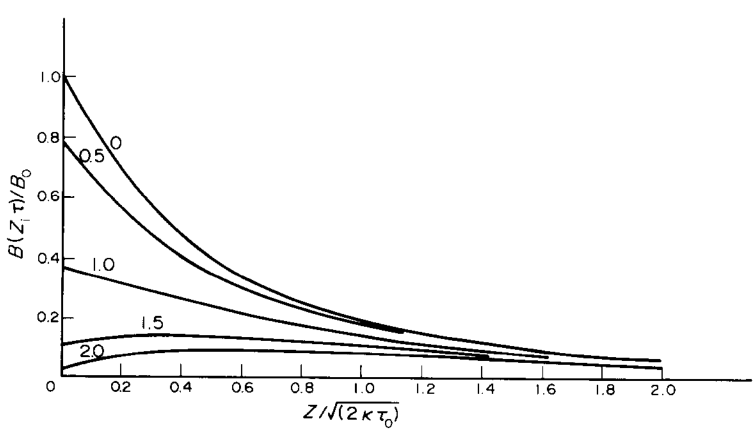

FIG. 2. Values of the temperature correlator $B(Z;\tau)$ in a semi-infinite medium. The boundary condition ($Z = 0$) is $B(0;\tau) = B_0 \exp(-\tau^2/\tau_0^2)$. Numbers on the curves denote the parameter τ/τ_0.

$$k^2 = \frac{4RR'}{(Z_1+Z_2)^2+(R+R')^2},$$
$$E(k) = \int_0^{\pi/2} d\phi\sqrt{(1-k^2\sin^2\phi)}. \quad (39)$$

$E(k)$ is the second-kind elliptic integral which decreases monotonically from the value $\pi/2$ at $k = 0$ to the value 1 at $k = 1$. At $k \ll 1$, $E(k)$ is given by the asymptotic expression [2]

$$E(k) \approx \frac{\pi}{2}\left(1-\frac{k^2}{4}-\frac{3}{64}k^4\right). \quad (40)$$

In the interval $1 \geqslant k \gtrsim 0.8$

$$E(k) \approx 1+\frac{1}{2}\left(\Lambda-\frac{1}{2}\right)k'^2+\frac{3}{16}\left(\Lambda-\frac{13}{12}\right)k'^4,$$
$$\Lambda = \ln\frac{4}{k'}, \quad k' = \sqrt{(1-k^2)}. \quad (41)$$

In the case of $(Z_1+Z_2)^2+R^2 \gg (R')^2 \sim 1$, equation (41) yields

$$B(Z_1+Z_2;R) \approx \frac{(Z_1+Z_2)T_1}{[(Z_1+Z_2)^2+R^2]^{3/2}} \times\left(1-\frac{3T_3}{2T_1[(Z_1+Z_2)^2+R^2]}+\frac{27R^2T_3}{4T_1[(Z_1+Z_2)^2+R^2]^2}\right),$$
$$T_n = \int_0^\infty dR\, R^n B_c(0;0;R). \quad (42)$$

For $Z_1+Z_2 \gg R \gg 1$, the correlator $B(Z_1+Z_2;R) \approx T_1(Z_1+Z_2)^{-2}$, i.e. it also decreases more slowly than in the presence of both spatial and temporal flucations. For $R \gg (Z_1+Z_2) \gg 1$, $B(Z_1+Z_2;R) \approx T_1 (Z_1+Z_2)R^{-3}$, i.e. the correlator $B(Z_1+Z_2;R)$ even increases with increasing Z_1+Z_2, since the entire region becomes occupied by the thermal wave which is correlated in a wide region of the order of R.

The diffusion process is described by the same equation as that for the temperature propagation, i.e. equation (2), if $T(\mathbf{r};t)$ is replaced by the number density of diffusing matter, $c(\mathbf{r};t)$, and the thermal diffusivity κ is replaced by the diffusion coefficient D. Determine the value of $c = c_0+c_1$, where $c_0(\mathbf{r};t) = \langle c(\mathbf{r};t)\rangle$ is the solution for an averaged equation of the form of equation (2) and $\langle c_1(\mathbf{r};t)\rangle = 0$. Then, all the equations obtained for the case of temperature correlation remain valid for the function $\langle c_1(1)c_1(2)\rangle$. For example, it is possible to solve the diffusion problem in the case of matter randomly falling on the surface of a medium in the form of drops having the characteristic transverse size R_0 and some time of contact with the surface τ_0. Then the intensity of fluctuations is described by the expression [see equation (23)]

$$\langle c_1^2(z;0)\rangle \approx \frac{1}{2\sqrt{\pi}}\cdot\frac{D\tau_0}{R_0^2}\left(\frac{R_0}{z}\right)^4\langle c_1^2(0;0)\rangle. \quad (43)$$

Similar formulae can be derived in the same manner.

It should be noted that all the formulae derived are suitable for both heat transfer and diffusion in a turbulent medium (which is at rest as a whole) if the characteristic scales R_0 and times τ_0 are much above the characteristic scales $R_{\text{turb.}}$ and times $\tau_{\text{turb.}}$ of turbulent pulsations. In this case the turbulent thermal diffusivity $\kappa_{\text{turb.}}$ and turbulent diffusion coefficient $D_{\text{turb.}}$ should be taken instead of κ and D, respectively.

The method applied can be easily extended to the case when the surface correlator $B(z_1;z_2;r_1;\phi_1;r_2;\phi_2;\tau)$ has a general form. Then the correlator $B(z_1;z_2;r_1;\phi_1;r_2;\phi_2;\tau)$ can be written as

$$B(1;2) = \sum_{m,n=0}^{\infty}\int_0^\infty d\lambda_1\,\lambda_1\int_0^\infty d\lambda_2\,\lambda_2 J_n(\lambda_1 r_1) \times J_m(\lambda_2 r_2)[\cos n\phi_1\cos m\phi_2 B_{nm}^{(1)} \times(z_1;z_2;\lambda_1;\lambda_2;\tau)+\cos n\phi_1\sin m \times\phi_2 B_{nm}^{(2)}(z_1;z_2;\lambda_1;\lambda_2;\tau)+\sin n\phi_1 \times\cos m\phi_2 B_{nm}^{(3)}(z_1;z_2;\lambda_1;\lambda_2;\tau) +\sin n\phi_1\sin m\phi_2 B_{nm}^{(4)}(z_1;z_2;\lambda_1;\lambda_2;\tau)]. \quad (44)$$

The functions $B_{nm}^{(i)}(z_1;z_2;\lambda_1;\lambda_2;\tau)$ are described by the equations which are analogous to equation (5)

$$\left[\frac{\partial}{\partial\tau}-\frac{\kappa}{2}\left(\frac{d^2}{dz_1^2}-\frac{d^2}{dz_2^2}-\lambda_1^2+\lambda_2^2\right)\right] \times B_{nm}^{(i)}(z_1;z_2;\lambda_1;\lambda_2;\tau) = 0$$
$$\left(\frac{d^2}{dz_1^2}+\frac{d^2}{dz_2^2}-\lambda_1^2-\lambda_2^2\right)B_{nm}^{(i)}(z_1;z_2;\lambda_1;\lambda_2;\tau) = 0. \quad (45)$$

These equations can be solved in the same way as equation (5). In particular, equation (9) holds for $s_{1,2}(\lambda_1)$ with the variable z_1 and for $s_{1,2}(\lambda_2)$ with z_2.

REFERENCES

1. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*. Clarendon Press, Oxford (1959).
2. E. Janke, F. Emde and F. Lösch, *Tafeln höherer Functionen*. B. G. Teuhner Verlaggesellschaft, Stuttgart (1960).

TRANSFERT THERMIQUE ET DIFFUSION A PARTIR DE SOURCES AVEC COMPOSANTES ALEATOIRES

Résumé—On considère le transfert thermique dans un milieu avec des surfaces chauffées au hasard. Des expressions explicites sont établies pour des formules de température en différents points du milieu. Des résultats analogues sont obtenus pour le problème de la diffusion dans le cas de sources avec des composantes aléatoires. Dans beaucoup de cas, quand la distance à la surface augmente, les corrélations diminuent comme la quatrième puissance de la profondeur.

WÄRMETRANSPORT UND DIFFUSION AUS ZUFÄLLIG VERTEILTEN QUELLGEBIETEN

Zusammenfassung—Der Wärmetransport in einem Körper wird betrachtet, dessen Oberfläche an zufälligen Stellen beheizt ist. Dies führt zu Temperaturkorrelationen an verschiedenen Stellen des Körpers. Analoge Resultate ergeben sich für das Diffusionsproblem aus zufällig verteilten Quellgebieten heraus. In vielen wichtigen Fällen verhalten sich die Ergebnisse umgekehrt proportional zur vierten Potenz der Entfernung von der Oberfläche.

ТЕПЛОПЕРЕНОС И ДИФФУЗИЯ ОТ ИСТОЧНИКОВ СО СЛУЧАЙНОЙ СОСТАВЛЯЮЩЕЙ

Аннотация—Рассмотрен перенос тепла и диффузия частиц для случая, когда поверхностные источники являются стохастическими. Подробно разобран случай полубесконечной среды. Показано, что в ряде случаев флуктуации температуры или концентрации затухают обратно пропорционально четвертой степени расстояния от стохастически нагреваемой поверхности.